ECE 174 HW #1 Solutions

Note Title 4/13/2009

Do the following 10 problems from the Meyer textbook:

1.2.8; 1.2.9; 2.1.3; 3.5.1; 3.6.3; 3.6.7; 3.7.5; 3.7.6; 3.7.8; 3.8.1

1.2.8. The following system has no solution:

$$\begin{aligned} -x_1 + 3x_2 - 2x_3 &= 1, \\ -x_1 + 4x_2 - 3x_3 &= 0, \\ -x_1 + 5x_2 - 4x_3 &= 0. \end{aligned}$$

Soln: Attempt to solve this system using Gaussian elimination and explain what occurs to indicate that the system is impossible to solve.

The third equation in the triangularized form is $0x_3 = 1$, which is impossible to solve.

$$\begin{pmatrix} -1 & 3 & -2 \\ -1 & 4 & -3 \\ -1 & 5 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & -3 & 2 & -1 \\ -1 & 4 & -3 & 0 \\ -1 & 5 & -4 & 0 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & -3 & 2 & -1 \\ 0 & 1 & -1 & -1 \\ 0 & 2 & -2 & -1 \end{array}\right)$$

$$= \left(\begin{array}{ccc|c} 1 & -3 & 2 & -1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

Impossible to solve!

1.2.9. Attempt to solve the system

$$\begin{aligned} -x_1 + 3x_2 - 2x_3 &= 4, \\ -x_1 + 4x_2 - 3x_3 &= 5, \\ -x_1 + 5x_2 - 4x_3 &= 6, \end{aligned}$$

using Gaussian elimination and explain why this system must have infinitely many solutions.

Soln The third equation in the triangularized form is $0x_3 = 0$, and all numbers are solutions. This means that you can start the back substitution with any value whatsoever and consequently produce infinitely many solutions for the system.

$$\left(\begin{array}{ccc|c} -1 & 3 & -2 & 4 \\ -1 & 4 & -3 & 5 \\ -1 & 5 & -4 & 6 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & -3 & 2 & -4 \\ -1 & 4 & -3 & 5 \\ -1 & 5 & -4 & 6 \end{array}\right)$$

$$= \left(\begin{array}{ccc|c} 1 & -3 & 2 & -4 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & -2 & 2 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & -3 & 2 & -4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Downarrow$$

$$0x_3 = 0$$

No restriction on values for x_3!

x_1, x_2 have restrictions once x_3 is chosen.

2.1.3. Suppose that $\mathbf{A}$ is an $m \times n$ matrix. Give a short explanation of why each of the following statements is true.

(a) $rank(\mathbf{A}) \leq \min\{m, n\}$.
(b) $rank(\mathbf{A}) < m$ if one row in $\mathbf{A}$ is entirely zero.
(c) $rank(\mathbf{A}) < m$ if one row in $\mathbf{A}$ is a multiple of another row.
(d) $rank(\mathbf{A}) < m$ if one row in $\mathbf{A}$ is a combination of other rows.
(e) $rank(\mathbf{A}) < n$ if one column in $\mathbf{A}$ is entirely zero.

Soln (a) Since any row or column can contain at most one pivot, the number of pivots cannot exceed the number of rows nor the number of columns. (b) A zero row cannot contain a pivot. (c) If one row is a multiple of another, then one of them can be annihilated by the other to produce a zero row. Now the result of the previous part applies. (d) One row can be annihilated by the associated combination of row operations. (e) If a column is zero, then there are fewer than n basic columns because each basic column must contain a pivot.

3.5.1. For $\mathbf{A} = \begin{pmatrix} 1 & -2 & 3 \\ 0 & -5 & 4 \\ 4 & -3 & 8 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & 2 \\ 0 & 4 \\ 3 & 7 \end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, compute the following products when possible.

(a) $\mathbf{AB}$, (b) $\mathbf{BA}$, (c) $\mathbf{CB}$, (d) $\mathbf{C}^T\mathbf{B}$, (e) $\mathbf{A}^2$, (f) $\mathbf{B}^2$,
(g) $\mathbf{C}^T\mathbf{C}$, (h) $\mathbf{CC}^T$, (i) $\mathbf{BB}^T$, (j) $\mathbf{B}^T\mathbf{B}$, (k) $\mathbf{C}^T\mathbf{AC}$.

Soln (a) $\mathbf{AB} = \begin{pmatrix} 10 & 15 \\ 12 & 8 \\ 28 & 52 \end{pmatrix}$ (b) $\mathbf{BA}$ does not exist (c) $\mathbf{CB}$ does not exist

(d) $\mathbf{C}^T\mathbf{B} = (10 \quad 31)$ (e) $\mathbf{A}^2 = \begin{pmatrix} 13 & -1 & 19 \\ 16 & 13 & 12 \\ 36 & -17 & 64 \end{pmatrix}$ (f) $\mathbf{B}^2$ does not exist

(g) $\mathbf{C}^T\mathbf{C} = 14$ (h) $\mathbf{CC}^T = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}$ (i) $\mathbf{BB}^T = \begin{pmatrix} 5 & 8 & 17 \\ 8 & 16 & 28 \\ 17 & 28 & 58 \end{pmatrix}$

(j) $\mathbf{B}^T\mathbf{B} = \begin{pmatrix} 10 & 23 \\ 23 & 69 \end{pmatrix}$ (k) $\mathbf{C}^T\mathbf{AC} = 76$

a) $AB = \begin{pmatrix} 1 & -2 & 3 \\ 0 & -5 & 4 \\ 4 & -3 & 8 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 4 \\ 3 & 7 \end{pmatrix}$

$= \begin{pmatrix} 1(1) + -2(0) + 3(3) & 1(2) + -2(4) + 3(7) \\ 0(1) + -5(0) + 4(3) & 0(2) + -5(4) + 4(7) \\ 4(1) + -3(0) + 8(3) & 4(2) + -3(4) + 8(7) \end{pmatrix}$

$= \begin{pmatrix} 10 & 15 \\ 12 & 8 \\ 28 & 52 \end{pmatrix}$

b) $BA = \begin{pmatrix} 1 & 2 \\ 0 & 4 \\ 3 & 7 \end{pmatrix} \begin{pmatrix} 1 & -2 & 3 \\ 0 & -5 & 4 \\ 4 & -3 & 8 \end{pmatrix} =$ dimension mismatch!

e) $A^2 = AA = \begin{pmatrix} 1 & -2 & 3 \\ 0 & -5 & 4 \\ 4 & -3 & 8 \end{pmatrix} \begin{pmatrix} 1 & -2 & 3 \\ 0 & -5 & 4 \\ 4 & -3 & 8 \end{pmatrix}$

g) $C^TC = (1 \quad 2 \quad 3) \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 1(1) + 2(2) + 3(3) = 14$

3.6.3. For the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 1 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 1 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 0 & 1/3 & 1/3 & 1/3 \end{pmatrix},$$

determine $\mathbf{A}^{300}$. **Hint:** A square matrix $\mathbf{C}$ is said to be *idempotent* when it has the property that $\mathbf{C}^2 = \mathbf{C}$. Make use of idempotent submatrices in $\mathbf{A}$.

Soln Partition the matrix as $\mathbf{A} = \begin{pmatrix} \mathbf{I} & \mathbf{C} \\ \mathbf{0} & \mathbf{C} \end{pmatrix}$, where $\mathbf{C} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ and observe that $\mathbf{C}^2 = \mathbf{C}$. Use this together with block multiplication to conclude that

$$\mathbf{A}^k = \begin{pmatrix} \mathbf{I} & \mathbf{C} + \mathbf{C}^2 + \mathbf{C}^3 + \cdots + \mathbf{C}^k \\ \mathbf{0} & \mathbf{C}^k \end{pmatrix} = \begin{pmatrix} \mathbf{I} & k\mathbf{C} \\ \mathbf{0} & \mathbf{C} \end{pmatrix}.$$

Therefore, $\mathbf{A}^{300} = \begin{pmatrix} 1 & 0 & 0 & 100 & 100 & 100 \\ 0 & 1 & 0 & 100 & 100 & 100 \\ 0 & 0 & 1 & 100 & 100 & 100 \\ 0 & 0 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 0 & 1/3 & 1/3 & 1/3 \end{pmatrix}.$

$$A^2 = \begin{pmatrix} I & C \\ 0 & C \end{pmatrix}\begin{pmatrix} I & C \\ 0 & C \end{pmatrix} = \begin{pmatrix} I & C+C^2 \\ 0 & C^2 \end{pmatrix}$$

$$= \begin{pmatrix} I & C+C \\ 0 & C \end{pmatrix} = \begin{pmatrix} I & 2C \\ 0 & C \end{pmatrix}$$

$$A^3 = A^2 A = \begin{pmatrix} I & 2C \\ 0 & C \end{pmatrix}\begin{pmatrix} I & C \\ 0 & C \end{pmatrix} = \begin{pmatrix} I & C+2C^2 \\ 0 & C^2 \end{pmatrix}$$

$$= \begin{pmatrix} I & C+2C \\ 0 & C \end{pmatrix} = \begin{pmatrix} I & 3C \\ 0 & C \end{pmatrix}$$

By induction,

$$A^k = \begin{pmatrix} I & kC \\ 0 & C \end{pmatrix}$$

3.6.7. For each matrix $\mathbf{A}_{n\times n}$, explain why it is impossible to find a solution for $\mathbf{X}_{n\times n}$ in the matrix equation

$$\mathbf{AX} - \mathbf{XA} = \mathbf{I}.$$

Hint: Consider the trace function.

Soln If a matrix $\mathbf{X}$ did indeed exist, then

$$\mathbf{I} = \mathbf{AX} - \mathbf{XA} \implies \mathit{trace}\,(\mathbf{I}) = \mathit{trace}\,(\mathbf{AX} - \mathbf{XA})$$
$$\implies n = \mathit{trace}\,(\mathbf{AX}) - \mathit{trace}\,(\mathbf{XA}) = 0,$$

Here you must realize that
trace$(A-B)$ = trace(A) − trace(B)
and trace(AX) = trace(XA)

3.7.5. If $\mathbf{A}$ is nonsingular and symmetric, prove that $\mathbf{A}^{-1}$ is symmetric.

= "has inverse" or "is invertible" . $A = A^T$

Soln $(\mathbf{A}^{-1})^T = (\mathbf{A}^T)^{-1} = \mathbf{A}^{-1}.$

Eqn. 3.7.16

3.7.6. If $\mathbf{A}$ is a square matrix such that $\mathbf{I} - \mathbf{A}$ is nonsingular, prove that

$$\mathbf{A}(\mathbf{I}-\mathbf{A})^{-1} = (\mathbf{I}-\mathbf{A})^{-1}\mathbf{A}.$$

Start with $\mathbf{A}(\mathbf{I}-\mathbf{A}) = (\mathbf{I}-\mathbf{A})\mathbf{A}$ and apply $(\mathbf{I}-\mathbf{A})^{-1}$ to both sides, first on one side and then on the other.

$A(I-A) = (I-A)A \iff AI - A^2 = IA - A^2$ starting point is true!

$(I-A)^{-1}A(I-A) = \cancel{(I-A)^{-1}}\cancel{(I-A)}A$

$(I-A)^{-1}A\cancel{(I-A)}\cancel{(I-A)^{-1}} = A(I-A)^{-1}$

$(I-A)^{-1}A = A(I-A)^{-1}$ ✓

3.7.8. If $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{A}+\mathbf{B}$ are each nonsingular, prove that

$$\mathbf{A}(\mathbf{A}+\mathbf{B})^{-1}\mathbf{B} = \mathbf{B}(\mathbf{A}+\mathbf{B})^{-1}\mathbf{A} = \left(\mathbf{A}^{-1}+\mathbf{B}^{-1}\right)^{-1}.$$

Soln Use the reverse order law for inversion to write

$$\left[\mathbf{A}(\mathbf{A}+\mathbf{B})^{-1}\mathbf{B}\right]^{-1} = \mathbf{B}^{-1}(\mathbf{A}+\mathbf{B})\mathbf{A}^{-1} = \mathbf{B}^{-1}+\mathbf{A}^{-1}$$

and

$$\left[\mathbf{B}(\mathbf{A}+\mathbf{B})^{-1}\mathbf{A}\right]^{-1} = \mathbf{A}^{-1}(\mathbf{A}+\mathbf{B})\mathbf{B}^{-1} = \mathbf{B}^{-1}+\mathbf{A}^{-1}.$$

3.8.1. Suppose you are given that

$$\mathbf{A} = \begin{pmatrix} 2 & 0 & -1 \\ -1 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{A}^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 2 \end{pmatrix}.$$

(a) Use the Sherman–Morrison formula to determine the inverse of the matrix $\mathbf{B}$ that is obtained by changing the $(3,2)$-entry in $\mathbf{A}$ from 0 to 2.

(b) Let $\mathbf{C}$ be the matrix that agrees with $\mathbf{A}$ except that $c_{32} = 2$ and $c_{33} = 2$. Use the Sherman–Morrison formula to find $\mathbf{C}^{-1}$.

Soln (a) $\mathbf{B}^{-1} = \begin{pmatrix} 1 & 2 & -1 \\ 0 & -1 & 1 \\ 1 & 4 & -2 \end{pmatrix}$

(b) Let $\mathbf{c} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and $\mathbf{d}^T = (0 \quad 2 \quad 1)$ to obtain $\mathbf{C}^{-1} = \begin{pmatrix} 0 & -2 & 1 \\ 1 & 3 & -1 \\ -1 & -4 & 2 \end{pmatrix}$

Sherman–Morrison Formula

- If $\mathbf{A}_{n\times n}$ is nonsingular and if $\mathbf{c}$ and $\mathbf{d}$ are $n \times 1$ columns such that $1 + \mathbf{d}^T\mathbf{A}^{-1}\mathbf{c} \neq 0$, then the sum $\mathbf{A} + \mathbf{c}\mathbf{d}^T$ is nonsingular, and

$$\left(\mathbf{A} + \mathbf{c}\mathbf{d}^T\right)^{-1} = \mathbf{A}^{-1} - \frac{\mathbf{A}^{-1}\mathbf{c}\mathbf{d}^T\mathbf{A}^{-1}}{1 + \mathbf{d}^T\mathbf{A}^{-1}\mathbf{c}}. \qquad (3.8.2)$$

a) $B = \begin{pmatrix} 2 & 0 & -1 \\ -1 & 1 & 1 \\ -1 & \boxed{2} & 1 \end{pmatrix}$

$$B = A + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} = A + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{c} \underbrace{(0 \ 2 \ 0)}_{d^T}$$

$$B^{-1} = (A + cd^T)^{-1} = A^{-1} - \frac{A^{-1}cd^TA^{-1}}{1 + d^TA^{-1}c}$$

$$= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 2 \end{pmatrix} - \frac{\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}(0 \ 2 \ 0)\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 2 \end{pmatrix}}{1 + (0 \ 2 \ 0)\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}$$

$$= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 2 \end{pmatrix} - \frac{\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}(0 \ 2 \ -2)}{1 + (0 \ 2 \ -2)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 2 \end{pmatrix} - \frac{\begin{pmatrix} 0 & 2 & -2 \\ 0 & -2 & 2 \\ 0 & 4 & -4 \end{pmatrix}}{\underbrace{1 + (-2)}_{= -1}}$$

$$= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 2 \end{pmatrix} + \begin{pmatrix} 0 & 2 & -2 \\ 0 & -2 & 2 \\ 0 & 4 & -4 \end{pmatrix} = \boxed{\begin{pmatrix} 1 & 2 & -1 \\ 0 & -1 & 1 \\ 1 & 4 & -2 \end{pmatrix}}$$

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b) $c = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad d^T = \begin{pmatrix} 0 & 2 & 1 \end{pmatrix} \Rightarrow cd^T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 1 \end{pmatrix}$

$$C^{-1} = (A + cd^T)^{-1} = A^{-1} - \frac{A^{-1} c d^T A^{-1}}{1 + d^T A^{-1} c}$$

$$= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 2 \end{pmatrix} - \frac{\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 & 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 2 \end{pmatrix}}{1 + \begin{pmatrix} 0 & 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}$$

$$= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 2 \end{pmatrix} - \frac{\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 0 \end{pmatrix}}{\underbrace{1 + \begin{pmatrix} 1 & 2 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{= 1 + 0}}$$

$$= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 2 & 0 \\ -1 & -2 & 0 \\ 2 & 4 & 0 \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} 0 & -2 & 1 \\ 1 & 3 & -1 \\ -1 & -4 & 2 \end{pmatrix}}$$